STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

2023

HURLSTONE AGRICULTURAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- NESA approved calculators may be used
- A reference sheet is provided
- For questions in Section II, use the relevant booklet for writing your solutions.
- This examination paper is not to be removed from the examination centre

Total marks: 70

Section I – 10 marks (pages 2 – 4)

- Attempt Questions 1 – 10.
- A multiple-choice answer sheet has been provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5 – 10)

- Attempt Questions 11 – 14, write your solutions in the booklets provided.
- You have been provided with 4 separate answer booklets, one for each question.
- Extra working pages are available if required.
- Allow about 1 hour and 45 minutes for this section.

SECTION I (10 marks)

Attempt Questions 1 - 10

Allow about 15 minutes on this section.

Use the multiple-choice answer sheet provided for Questions 1 – 10.

1. The equation $\frac{dP}{dt} = 0.4P(5000 - P)$ is used to describe the population of kangaroos in a national park.

   What is the maximum population of kangaroos, according to this model?

   (A) 50 (B) 2000

   (C) 2500 (D) 5000

2. Given that $P = 20 + 3980e^{-kt}$ and $P = 3119$ when $t = 1$, what is the value of k?

   (A) 0.25 (B) –0.25

   (C) 0.10 (D) –0.10

3. Let S be the surface area of a sphere with radius r.

   What is the exact value of $\frac{dr}{dS}$ when $r = 7$ units?

   (A) 56π (B) $\frac{7}{8\pi}$

   (C) $\frac{8}{7\pi}$ (D) $\frac{1}{56\pi}$

4. A vector $\overrightarrow{AB}$ has a magnitude of 10. When its tail is at origin it lies between the positive x −axis and positive y −axis and makes anlge of 60^0 with the x − axis.

   Which of the following is vector $\overrightarrow{BA}$?

   (A) $5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}$ (B) $-5\sqrt{3}\underset{\sim}{i} - 5\underset{\sim}{j}$

   (C) $5\underset{\sim}{i} + 5\sqrt{3}\underset{\sim}{j}$ (D) $-5\underset{\sim}{i} - 5\sqrt{3}\underset{\sim}{j}$

5. A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ has real roots at x_1, x_2 and x_3, which are all **distinct** values of x .
Given that $P'(x_1) = 0$, $P'(x_2) = P''(x_2) = 0$ and $P'(x_3) = P''(x_3) = P'''(x_3) = 0$, what is the lowest possible degree of $P(x)$?

(A) $n = 3$

(B) $n = 6$

(C) $n = 7$

(D) $n = 9$

---

6. The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

What are the possible values of a?

(A) $a = -2$ or $a = 3$

(B) $a = -1$ or $a = 6$

(C) $a = 2$ or $a = -3$

(D) $a = 1$ or $a = -6$

---

7. Which of the following is the derivative of $\tan^{-1}(3x)$?

(A) $3\tan^{-1}x$

(B) $\frac{3}{1+x^2}$

(C) $\frac{3}{1+9x^2}$

(D) $3\sec^2(3x)$

---

8. A Bernoulli variable, X, has a value of p such that $E(X) = 5Var(X)$.
Given that $p \neq 0$, what is the value of p?

(A) $\frac{1}{2}$

(B) $\frac{4}{5}$

(C) $\frac{1}{5}$

(D) $\frac{3}{5}$

---

9. What is the coefficient of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^5$?

(A) 20 (B) 40

(C) 120 (D) 240

10. Out of a group of 10 professional tiddlywinks players, 6 are to be selected for the final round of a competition. 4 of those 6 will be placed as 1st, 2nd, 3rd and 4th.

In how many ways can this process be carried out?

(A) $\frac{10!}{6!4!}$ (B) $\frac{10!}{4!4!}$

(C) $\frac{10!}{4!2!}$ (D) $\frac{10!}{6!}$

~ End of Section 1 ~

SECTION II

60 marks

Attempt Questions 11 - 14

Allow about 1 hours and 45 minutes on this section.
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.
Answer each question in a separate answer booklet.

Question 11 (15 marks) Use the Question 11 BOOKLET **Marks**

(a) Water at a temperature of $24°\text{C}$ is placed in a freezer maintained at a temperature of $-12°\text{C}$. After time *t* minutes the rate of change of temperature *T* of the water is given by the formula:

$$\frac{dT}{dt} = -k(T + 12)$$

where *k* is a positive constant.

(i) Show that $T = Ae^{-kt} - 12$ is a solution of this equation, where *A* is a constant. **1**

(ii) Find the value of *A*. **1**

(iii) After 20 minutes the temperature of the water falls to 6°C. Find to the nearest minute the time taken for the water to start freezing.
(Freezing point of water is 0°C). **3**

(b) A population of 1200 feral cats is released into an area.

The rate of increase of the feral cat population is: $\frac{dP}{dt} = \frac{P}{30}\left(1 - \frac{P}{10000}\right)$,
where *P* is the feral cat population and *t* is the number of months after release.

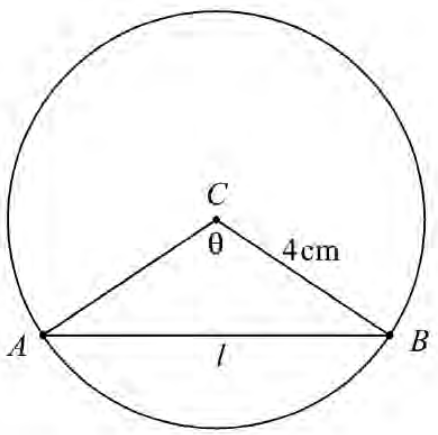
(i) Given that $\frac{10000}{P(10000-P)} = \frac{1}{P} + \frac{1}{10000-P}$ **(you do not need to prove this)**,
solve the differential equation: $\frac{dP}{dt} = \frac{P}{30}\left(1 - \frac{P}{10000}\right)$. **2**

(ii) Hence, find the limiting feral cat population. **1**

Question 11 is continued on next page …

Question 11 continued … **Marks**

(c) In the diagram, *C* is the centre of a circle of radius 4 cm and $\angle ACB = \theta$ radians.



The length of chord *AB* is *l* cm.

(i) Show that the length *l* of *AB* is given by $l = 4\sqrt{2 - 2\cos\theta}$. **1**

(ii) If θ is increasing at the rate of $\frac{2}{3}$ rad s^{-1}, find the rate of change of the length of *AB* when $\theta = \frac{\pi}{3}$ radians. Express your answer in simplified exact form. **3**

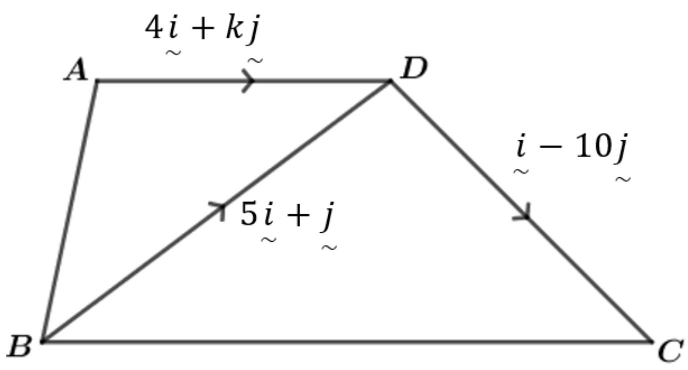
(d) Prove the following by using Mathematical Induction for integer values of *n*.

$$(2^1 + 2) + (2^2 + 4) + (2^3 + 6) + \cdots + (2^n + 2n) = 2^{n+1} + n(n+1) - 2, \text{ for } n \geq 1$$ **3**

~ End of Question 11 ~

Question 12 (15 marks) Use the Question 12 BOOKLET **Marks**

(a) The figure below shows a trapezium *ABCD*, where *AD* is parallel to *BC*.



The following information is given for this trapezium:

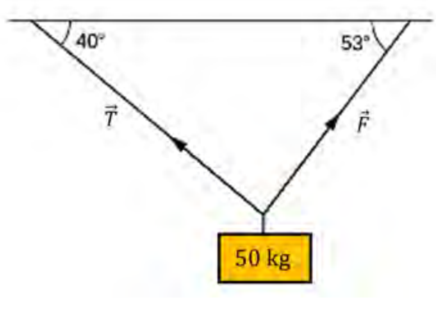
$\overrightarrow{BD} = 5\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{DC} = \underset{\sim}{i} - 10\underset{\sim}{j}$ and $\overrightarrow{AD} = 4\underset{\sim}{i} + k\underset{\sim}{j}$, where *k* is an integer.

(i) Use vector algebra to find the value of *k*. **2**

(ii) Find the length of $\overrightarrow{AB}$. **1**

(iii) Calculate the size of angle *ABD*. **2**

(b) A 50 kg weight is hung by a cable so that the two portions of the cable make angles of 40^o and 53^o, respectively, with the horizontal as shown below.



Find the magnitudes of the tension forces $\vec{T}$ and $\vec{F}$ in the cables if the resultant force acting on the object is zero. Give your answer in Newtons by rounding your answer to two decimal places. (use $g = 9.8$) **3**

Question 12 is continued on next page …

Question 12 continued … **Marks**

(c) The equation $x^3 - kx^2 + (k-1)x - (k+2) = 0$ has roots α, β and γ .

(i) Show that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{k-1}{k+2}$ **2**

(ii) Find the set of values of k such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq 0$ **2**

(d) Show that $(n+1)[n!\,n + (n-1)!\,(2n-1) + (n-2)!\,(n-1)] = (n+2)!$ **3**

~ End of Question 12 ~

Question 13 (15 marks) Use the Question 13 BOOKLET **Marks**

(a) Consider the function $f(x) = x^2 - 4x + 6$.

(i) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(ii) Given that the domain of $f(x)$ is restricted to $x \geq 2$, find an expression for $f^{-1}(x)$. **2**

(iii) Given the restriction in part (ii), sketch $y = f^{-1}(x)$. **2**

(iv) Find all points of intersection with the curve $y = f(x)$, and the curve $y = f^{-1}(x)$. **2**

(b) By use of a suitable sketch, or otherwise, solve $\sin^{-1}(3x + 1) = \cos^{-1}x$. **3**

(c) In a certain school, 23% of Year 12 students study Mathematics Extension 1.

(i) If X represents the number of students who study Mathematics Extension 1, describe the skewness of the binomial distribution for $P(X = x)$. **1**
You must give reasons for your answer.

(ii) The Principal meets with a random sample of 60 Year 12 students, to discuss their Year 12 HSC courses.
What is the probability that more than 30% of the students that meet with the Principal study Mathematics Extension 1?
(You may assume that the sample of students is approximately normally distributed, and make reference to the extract below from a table of z scores). **4**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

~ End of Question 13 ~

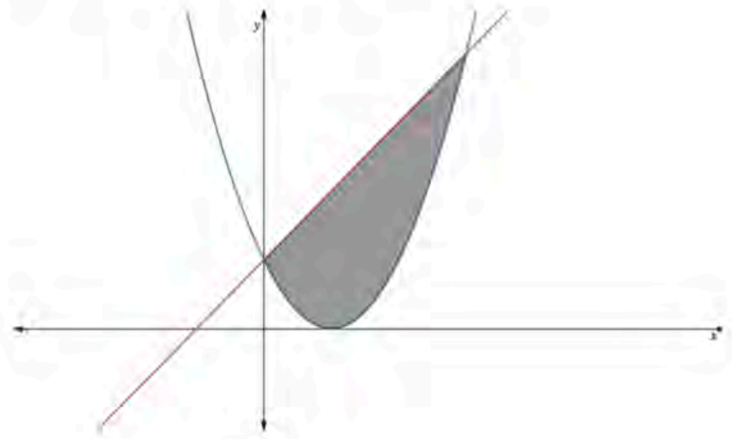
Question 14 (15 marks) Use the Question 14 BOOKLET **Marks**

(a) (i) Show that $\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$. **1**

(ii) Hence, find the exact value of $\int_0^{\sqrt{3}} \tan^{-1} x\, dx$. **2**

(b) The shaded area in the diagram below is bounded by $y = (x-1)^2$ and $y = x+1$.
The area is rotated about the x-axis to form a solid.

Find the volume of the solid that is generated. **4**



(c) (i) Show that the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2-1}$ describes a family of parabolas with x-intercepts at (1,0) and (-1,0) **2**

(ii) Now consider the differential equation $\frac{dy}{dx} = \frac{1-x^2}{2xy}$.
Find the equation of the curve which satisfies this differential equation and passes through the point (1,1). Express your answer as a function of x. **2**

(iii) What can be said about the gradient of the curve in part (ii) in relation to the family of curves in part (i)? **1**

(d) A committee of five is to be chosen from six men and seven women.

(i) How many committees are possible if there are no restrictions? **1**

(ii) How many committees are possible if there are more women than men? **2**

~ End of Question 14 ~

~ End of Trial Examination ~

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PAGE

Nothing to see here.

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \ r \neq 1$$

$$S = \frac{a}{1-r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

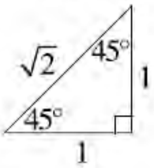
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

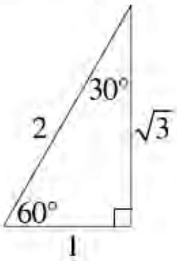
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

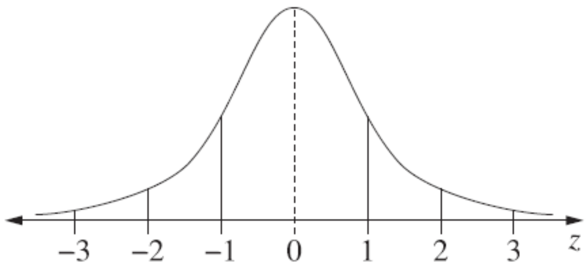
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

HURLSTONE AGRICULTURAL HIGH SCHOOL
2023 Trial Higher School Certificate Examination
Mathematics Extension 1

Name ______________________________ Teacher ______________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ● B ● C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Marking Guidelines Mathematics Extension 1 Trial Examination

Multiple Choice:

Question 1

Answer: D

Maximum can only occur when $\frac{dP}{dt} = 0$ The only non-zero answer is $P = 5000$

Question 2

Answer: A

Substitution and rearrangement gives: $-k = \ln\frac{3099}{3980} \rightarrow \ k \approx 0 \cdot 25.$

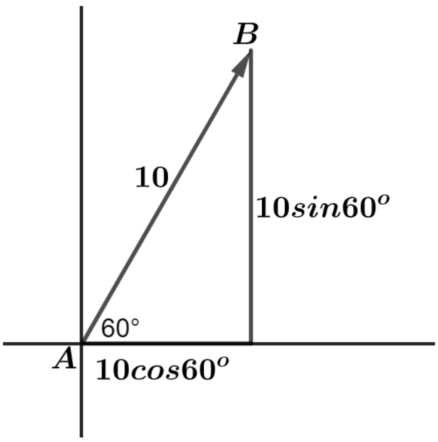
Question 3

Answer: D

$S = 4\pi r^2 \quad \rightarrow \quad \frac{dS}{dr} = 8\pi r \quad \rightarrow \quad \frac{dr}{dS} = \frac{1}{8\pi r} = \frac{1}{56\pi}$

Question 4

Answer: D



Horizontal component of $\overrightarrow{AB} = 10cos60^o = 5$

Vertical component of $\overrightarrow{AB} = 10sin60^o = 5\sqrt{3}$

$\overrightarrow{AB} = 5\underset{\sim}{i} + 5\sqrt{3}\underset{\sim}{j}$

$\therefore \overrightarrow{BA} = -5\underset{\sim}{i} - 5\sqrt{3}\underset{\sim}{j}$

Question 5

Answer D

$P'(x_1) = 0 \implies \therefore$ double zero at $x = x_1$ $\qquad$ $P''(x_2) = 0 \implies \therefore$ triple zero at $x = x_2$

$P'''(x_3) = 0 \implies \therefore$ quad zero at $x = x_3$

$\therefore n = 2 + 3 + 3 = 9$

Question 6

Answer: C

$$a^2e^{ax} + ae^{ax} - 6e^{ax} = 0$$

$$e^{ax}(a^2 + a - 6) = 0$$

$$e^{ax} \neq 0\,, \therefore\ (a+3)(a-2) = 0$$

Question 7

Answer: C

From reference sheet: $\frac{d}{dx}tan^{-1}(3x) = \frac{3}{1+(3x)^2}$

Question 8

Answer: B

Statement means: $np = 5npq \ \rightarrow\ q = \frac{1}{5} \ \rightarrow\ p = \frac{4}{5}$

Question 9

Answer B

The required term is $\binom{5}{3}(2x^2)^2\left(\frac{1}{x}\right)^3$

The coefficient is $\frac{5!}{3!2!} \times 2^2 = 40$

Question 10

Answer: C

First selection is of a group; second selection is an ordered one.

${}^{10}C_6$ followed by ${}^{6}P_4$: $\frac{10!}{6!4!} \times \frac{6!}{2!}$

<table>
<tr><th>Outcome</th><th>Solutions</th><th>Marking Guidelines</th></tr>
<tr><td>(a)

(b)</td><td>

(a) (i) $T = Ae^{-kt} - 12$

$\frac{dT}{dt} = -kAe^{-kt} = -k\left(Ae^{-kt} - 12 + 12\right)$

$= -k(T + 12)$ as required.

(ii) When $t = 0$, $T = 24$

$$\therefore 24 = Ae^{0} - 12$$
$$A = 36$$

(iii) When $t = 20$, $T = 6$

$$\therefore 6 = 36e^{-20k} - 12$$
$$\frac{1}{2} = e^{-20k}$$
$$k = \frac{-1}{20}\,ln\frac{1}{2} = \frac{ln2}{20}$$

Finding t when T=0

$$0 = 36e^{-kt} - 12$$
$$\frac{1}{3} = e^{-kt} \rightarrow -kt = ln\frac{1}{3}$$
$$t = -\frac{1}{k}ln\frac{1}{3} = \frac{1}{k}\,ln3 = \frac{20ln3}{ln2}$$
$t = 31 \cdot 699 \rightarrow t = 32$ min (nearest min)

(b) (i)

$$\frac{dP}{dt} = \frac{P}{30}\left(\frac{10000 - P}{10000}\right)$$
$$\therefore \frac{dt}{dP} = 30 \times \frac{10000}{P(10000 - P)}$$
$$= 30\left(\frac{1}{P} + \frac{1}{10000 - P}\right)$$
$$t = 30\int \frac{1}{P} + \frac{1}{10000 - P}\,dP$$
$$= 30\,ln\left|\frac{P}{10000 - P}\right| + c$$

Initial conditions: $0 = 30ln\frac{1200}{8800} + c \rightarrow c = -30\,ln\frac{3}{22}$

$$\therefore t = 30\ln\left|\frac{P}{10000 - P}\right| - 30ln\frac{3}{22}$$
$$\rightarrow P = \frac{30000e^{\frac{t}{30}}}{22 + 3e^{\frac{t}{30}}}$$

(ii)

$$P = 30000\left(\frac{1}{22e^{\frac{-t}{30}} + 3}\right)$$

As $t \rightarrow \infty$ $P \rightarrow 30000\left(\frac{1}{0+3}\right) = 10000$,which is the limiting population.

</td><td>

(a) (i) 1 mark: Correct solution.
Note: Other good solutions involved starting with the opposite statement and integrating. Both methods achieved “show that”.

(ii) 1 mark: Correct solution.

(iii) 3 marks: Correct solution showing expressions for k, kt and the final rounded answer.
2 marks: Almost fully correct solution.
1 mark: Considerable relevant progress.
Note: Best answers kept k as an exact value until the final calculation.

(b) (i) 2 marks: Complete solution, including correct primitive for t and making P the subject of equation in terms of t only.
1 mark: One aspect of the solution fully complete.

(ii) 1 mark: Correct solution must come from an expression from which a limit can be justified.
0 marks: Answer of 10000 without evidence in (ii) or (i) that limit can be justified. (i.e. e raised to a positive power does not approach a limit.)

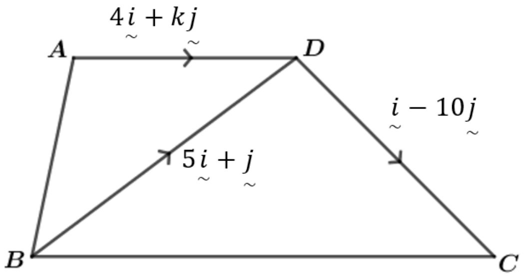
</td></tr>
</table>

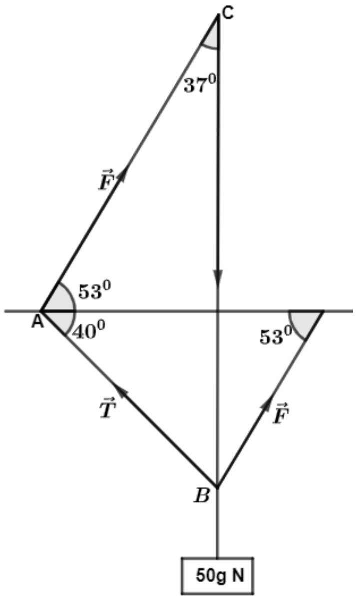
(c) **(d)**	**(c) (i)** Using the cosine rule: $l^2 = 4^2 + 4^2 - 2 \times 4^2 cos\theta$ $l^2 = 32 - 32cos\theta$ $\therefore l = 4\sqrt{2 - 2cos\theta}$ **(ii)** $\frac{dl}{dt} = \frac{dl}{d\theta} \times \frac{d\theta}{dt} = \frac{d}{d\theta}\left(4(2-2cos\theta)^{\frac{1}{2}}\right) \times \frac{2}{3}$ $= \left(\frac{1}{2}\right) 4(2-2cos\theta)^{-\frac{1}{2}}(2sin\theta)\frac{2}{3}$ $= \frac{8sin\theta}{3\sqrt{2-2cos\theta}}$ When $\theta = \frac{\pi}{3}$ $\frac{dl}{dt} = \frac{8\frac{\sqrt{3}}{2}}{3\sqrt{2-2\left(\frac{1}{2}\right)}} = \frac{4\sqrt{3}}{3}$ cm per second **(d)** Step 1: When $n = 1$, $LHS = 2^1 + 2(1) = 4$ $RHS = 2^2 + 1(2) - 2 = 4 + 2 - 2$ $= 4 = LHS$ Hence true when $n = 1$. Step 2: Assume $(2^1 + 2) + (2^2 + 4) + \cdots . + (2^k + 2k) = 2^{k+1} + k(k+1) - 2$ Prove: $(2^1 + 2) + (2^2 + 4) + \cdots . + 2^{k+1} + 2(k+1)$ $= 2^{k+2} + (k+1)(k+2) - 2$ $LHS = 2^{k+1} + k(k+1) - 2 + 2^{k+1} + 2(k+1)$ $= 2 \times 2^{k+1} + (k+1)(k+2) - 2$ $= 2^{k+2} + (k+1)(k+2) - 2$ $= RHS$ Statement proven by mathematical induction.	**(c) (i) 1 mark:** Correct solution. **(ii) 3 marks:** Correct solution. **2 marks:** Almost all aspects of solution correct. **1 mark: Correct application** of chain rule must be included in a partial solution. **(d) 3 marks:** Complete solution. **2 marks:** Almost complete solution with one detail of proof omitted. **1 mark:** Some relevant progress. *Note: After the "Prove for n=k+1" step is set up, note that the RHS still has a factor of (k+1) and a minus 2. Those who got "stuck" in the proof tended to expand or cancel these features and hence had trouble making LHS = RHS. Moral of the story: Look at the RHS so that you know the "destination" of your simplification.*

Year 12	Mathematics Extension 1 2023	TASK 4
Question No. 12	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

ME12-2 applies concepts and techniques involving vectors and projectiles to solve problems
ME11-1 uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses
ME11-2 manipulates algebraic expressions and graphical functions to solve problem
ME11-5 uses concepts of permutations and combinations to solve problems involving counting or ordering

Part / Outcome	Solutions	Marking Guidelines
		
(a)(i) **ME12-2**	Looking at the diagram $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$ $\overrightarrow{BC} = 5\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{i} - 10\underset{\sim}{j}$ $\overrightarrow{BC} = 6\underset{\sim}{i} - 9\underset{\sim}{j}$ As AD is parallel to BC, their vector components must be in proportion. $\frac{4}{k} = \frac{6}{-9}$ $6k = -36$ $k = -6$	**2 marks** – Correct solution **1 mark** – Substantially correct
(a)(ii) **ME12-2**	$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = 4\underset{\sim}{i} + k\underset{\sim}{j} - 5\underset{\sim}{i} + \underset{\sim}{j} = -\underset{\sim}{i} - 7\underset{\sim}{j}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{(-1)^2 + (-7)^2} = 5\sqrt{2}$	**1 mark**– Correct solution
(a)(iii) **ME12-2**	$\cos(\angle ABD) = \frac{\overrightarrow{BA}\cdot\overrightarrow{BD}}{\lvert\overrightarrow{BA}\rvert\lvert\overrightarrow{BD}\rvert}$ $\cos(\angle ABD) = \frac{(1\times 5) + (7\times 1)}{\left(\sqrt{(1)^2+(7)^2}\right)\left(\sqrt{(5)^2+(1)^2}\right)}$ $\cos(\angle ABD) = \frac{12}{(\sqrt{50})(\sqrt{26})}$ $\angle ABD = \cos^{-1}\frac{12}{\sqrt{1300}} = 70.5599\ldots \approx 70.6^0$	**2 marks** – Correct solution **1 mark** – Substantially correct

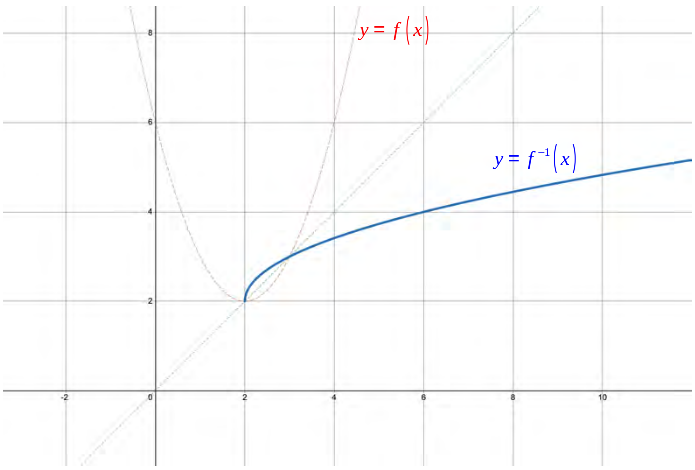
(b) **ME12-2**	 $50\ kg = 50g$ N Representing the forces in a triangle as shown above and using Sine rule in triangle ABC $\frac{\lvert\vec{T}\rvert}{\sin 37^0} = \frac{\lvert\vec{F}\rvert}{\sin 50^0} = \frac{50g}{\sin(53^0 + 40^0)}$ $\frac{\lvert\vec{T}\rvert}{\sin 37^0} = \frac{\lvert\vec{F}\rvert}{\sin 50^0} = \frac{50g}{\sin 93^0}$ $\frac{\lvert\vec{T}\rvert}{\sin 37^0} = \frac{50g}{\sin 93^0}$ $\frac{\lvert\vec{F}\rvert}{\sin 50^0} = \frac{50g}{\sin 93^0}$ $\lvert\vec{T}\rvert = \frac{50g}{\sin 93^0} \times \sin 37^0 = 295.29N$ $\lvert\vec{F}\rvert = \frac{50g}{\sin 93^0} \times \sin 50^0$=375.89N	**3 marks** – Correct solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution
(c)(i) **ME 11-2**	α, β, γ are roots of $x^3 - kx^2 + (k-1)x - (k+2) = 0$ $\therefore \alpha\beta\gamma = -\frac{d}{a} = -\big(-(k+2)\big) = k+2$ and $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = k - 1$ LHS= $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{k-1}{k+2}$ =RHS	**2 marks** – Correct solution **1 mark** – Substantially correct
(c)(ii) **ME 11-2**	$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq 0$ $\frac{k-1}{k+2} \geq 0$ $(k-1)(k+2) \geq 0$ $\therefore k < -2\ or\ k \geq 1$ (as $k \neq -2$)	**2 marks** – Correct solution **1 mark** – Substantially correct

(d) **ME 11-5**	LHS$=(n+1)[n!\,n+(n-1)!\,(2n-1)+(n-2)!\,(n-1)]$ $=(n+1)[n(n-1)!\,n+(n-1)!\,(2n-1)+(n-1)!]$ $=(n+1)[n^2(n-1)!+(n-1)!\,(2n-1)+(n-1)!]$ $=(n+1)(n-1)!\,[n^2+(2n-1)+1]$ $=(n+1)(n-1)!\,[n^2+2n]$ $=(n+1)(n-1)!\,n(n+2)$ $=(n+2)(n+1)n(n-1)!$ $=(n+2)!$ =RHS	**3 marks** – Correct solution **2 marks** – Substantially correct solution. **1 mark** – some correct working towards correct solution

Year 12	Mathematics Extension 1 2023	TASK 4
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

ME11-1 – uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses
ME11-3 – applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems
ME12-3 – applies advanced concepts and techniques in simplifying expressions involving compound angles and solving trigonometric equations
ME12-5 – applies appropriate statistical processes to present, analyse and interpret data

Part / Outcome	Solutions	Marking Guidelines
(a)(i) **ME11-1**	$f(x)=x^2-4x+6$ is a parabola, and <u>**fails the horizontal line test**</u>. So, the inverse of $f(x)$ over its entire domain is not a function.	**1 mark** – Correct solution (*explains using the horizontal line test, or one-to-one function, or equivalent merit*)
(a)(ii) **ME11-1**	Complete the square to express $f(x)$ in vertex form: $f(x)=x^2-4x+6 \quad (x\le 2)$ $=(x-2)^2+2$ Swap x and y, and make y the subject: $x=(y-2)^2+2 \quad (y\le 2)$ $x-2=(y-2)^2$ $y-2=-\sqrt{x-2} \quad (\text{discard } +\sqrt{x-2} \text{ as } y\le 2)$ $y=-\sqrt{x-2}+2$ $f^{-1}(x)=-\sqrt{x-2}+2$	**2 marks** – Correct solution **1 mark** – Substantially correct (*swaps x and y, <u>**and**</u> attempts to makes y the subject… or equivalent merit)*
(a)(iii) **ME11-1**		**2 marks** – Correct solution (note the vertical tangent at (2, 2)) **1 mark** – Substantially correct

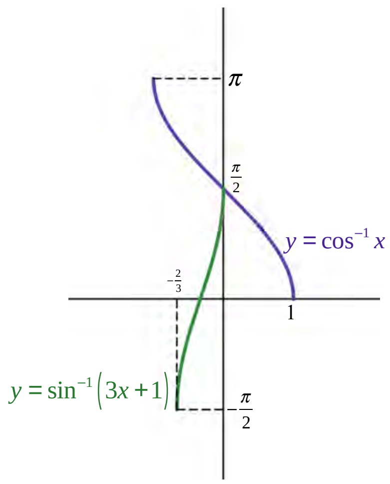
<table>
<tr><td></td><td>Question 13 continued…</td><td></td></tr>
<tr><td>(a)(iv)
ME11-1</td><td>Noting $y=f\left(x\right)$ and $y=f^{-1}\left(x\right)$ intersect at $y=x$,

solve $y=x$
and $y=x^2-4x+6$
so, $x=x^2-4x+6$
$x^2-5x+6=0$
$\left(x-3\right)\left(x-2\right)=0$
$x=2$ or 3
$\therefore$ the points of intersection are $\left(2,2\right)$ and $\left(3,3\right)$</td><td>2 marks – Correct solution

1 mark – significant progress towards correct solution
note: points not awarded if no point given
• x=2 is not a point
• (2, 2) should be obvious!</td></tr>
<tr><td>(b)
ME11-3
ME12-3</td><td>Noting
$y=\sin^{-1}\left(3x+1\right)$
$=\sin^{-1}\left(3\left(x+\frac{1}{3}\right)\right)$

ie dilation by $\frac{1}{3}$

shift by $-\frac{1}{3}$



$\therefore x=0$</td><td>3 marks – Correct solution

2 marks – significant progress towards correct solution

1 mark – partial progress towards correct solution

Note
• the solution is for x, NOT for y.
• suggested method (graphing) is best… many algebraic solutions hinted at extra solutions</td></tr>
<tr><td>(c)(i)
ME12-5</td><td>Since $p = 0.23 < 0.5$, where p is the probability of a student studying Maths Ext 1, the graph will have a positive skew (skewed to the right).</td><td>1 mark – Correct solution
(MUST give reason)</td></tr>
<tr><td>(c)(ii)
ME12-5</td><td>$\mu=p=0.23$

$\sigma^2=\frac{p\left(1-p\right)}{n}=\frac{0.23\times0.77}{60}$
$=0.0029516...$
$\sigma=0.054329...$

$z=\frac{x-\mu}{\sigma}=\frac{0.3-0.23}{0.054329}$
$=1.28844...$
$=1.29$

$P\left(z<1.29\right)=0.9015$
$P\left(z>1.29\right)=1-0.9015$
$=0.0985$</td><td>4 marks – Correct solution

3 marks – Substantially correct solution

2 marks – meaningful progress towards correct solution

1 mark – partial progress towards correct solution

(note: rounding too early gives incorrect/inaccurate solution)</td></tr>
</table>

2023 Yr12 HSC Assessment Task 4

Question 14 Solutions and Marking Guidelines

Outcomes Addressed in this Question

ME12-4: uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

ME12-5: applies appropriate statistical processes to present, analyse and interpret data

Solutions	Marking Guidelines
a i) $$\frac{d}{dx}(x\tan^{-1}x) = x \times \frac{1}{1+x^2} + 1 \times \tan^{-1}x$$ $$= \tan^{-1}x + \frac{x}{1+x^2}$$	1 mark Correct solution with all important steps shown
ii) $$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$$ $$d(x\tan^{-1}x) = \tan^{-1}x dx + \frac{x}{1+x^2}dx$$ $$\int_0^{\sqrt{3}} d(x\tan^{-1}x) = \int_0^{\sqrt{3}} \tan^{-1}x\,dx + \int_0^{\sqrt{3}} \frac{x}{1+x^2}dx$$ $$\int_0^{\sqrt{3}} \tan^{-1}x\,dx = \int_0^{\sqrt{3}} d(x\tan^{-1}x) - \int_0^{\sqrt{3}} \frac{x}{1+x^2}dx$$ $$= [x\tan^{-1}x]_0^{\sqrt{3}} - \left[\frac{1}{2}\ln\|1+x^2\|\right]_0^{\sqrt{3}}$$ $$= \sqrt{3}\tan^{-1}\sqrt{3} - 0 - \frac{1}{2}\ln\left\|1+\sqrt{3}^2\right\| + \frac{1}{2}\ln\|1+0^2\|$$ $$= \frac{\pi\sqrt{3}}{3} - \frac{\ln 4}{2}$$	2 marks Correct solution 1 mark Substantial progress to correct solution

b) Intersection x-coordinates: $$x+1=(x-1)^2$$ $$x^2-3x=0$$ $$x(x-3)=0$$ $$x=0,3$$ $$V=\pi\int_0^3 [(x+1)^2-[(1-x)^2]^2]\,dx$$ $$=\pi\left[\frac{(x+1)^3}{3}-\frac{(x-1)^5}{5}\right]_0^3$$ $$=\pi\left[\frac{(3+1)^3}{3}-\frac{(3-1)^5}{5}-\frac{(0+1)^3}{3}+\frac{(0-1)^5}{5}\right]$$ $$=\frac{72\pi}{5}$$	4 marks Correct solution 3 marks Correctly sets up integral to find volume 2 marks Substantial progress towards solution 1 mark Some progress towards solution.
C i) $$\frac{dy}{dx}=\frac{2xy}{x^2-1}$$ $$\int\frac{dy}{y}=\int\frac{2x}{x^2-1}dx$$ $$\ln\lvert y\rvert=\ln\lvert x^2-1\rvert+c$$ $$y=e^{\ln\lvert x^2-1\rvert+c}$$ $$y=(x^2-1)e^c \quad \text{note: } \mathrm{e}^{\mathrm{c}}=k$$ $$=k(x-1)(x+1)$$	2 marks Correct solution 1 mark substantial progress towards solution.
ii) $$\frac{dy}{dx}=\frac{1-x^2}{2xy}$$ $$\int y dy=\frac{1}{2}\int\left(\frac{1}{x}-x\right)dx$$ $$\frac{y^2}{2}=\frac{1}{2}\left(\ln\lvert x\rvert-\frac{x^2}{2}+c\right)$$ $$y^2=\ln\lvert x\rvert-\frac{x^2}{2}+c$$	2 marks Correct solution in mod-arg form 1 mark Substantial progress to correct solution

Passes through (1,1) $$1^2 = \ln\lvert 1\rvert - \frac{1^2}{2} + c$$ $$c = \frac{3}{2}$$ $$y^2 = \ln\lvert x\rvert - \frac{x^2}{2} + \frac{3}{2} \qquad \text{where } x \neq 0$$	
iii) multiplying the two differential equations, we have: $$\frac{2xy}{x^2-1} \times \frac{1-x^2}{2xy} = \frac{-(x-1)(x+1)}{(x-1)(x-1)}$$ $$= -1$$ Since the two are gradient functions and when multiplied the product is -1, the gradient of curve in part ii), that is the tangent, is always perpendicular to the family of curves in part i).	1 mark Valid explanation of relationship
d i) Committee of 5 from 6 males and 7 females $${}^{7+6}C_5 = 1287$$	1 mark Correct Solution
ii) 5 female → ${}^{7}C_5 = 21$ 4 female, 1 male → ${}^{7}C_4 + {}^{6}C_1 = 210$ 3 female, 2 male → ${}^{7}C_3 + {}^{6}C_2 = 526$ Total is $21 + 210 + 525 = 756$	2 marks Correct solution 1 mark Substantial progress to correct solution